Perturbing general uncorrelated networks

Z. Burda,[1] J. Jurkiewicz,[1] and A. Krzywicki[2]

[1]*M. Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, 30-059 Krakow, Poland*
[2]*Laboratoire de Physique Théorique, Bâtiment 210, Université Paris-Sud, 91405 Orsay, France*

This paper is a direct continuation of an earlier work, where we studied Erdös-Rényi random graphs perturbed by an interaction Hamiltonian favoring the formation of short cycles. Here, we generalize these results. We keep the same interaction Hamiltonian but let it act on general graphs with uncorrelated nodes and an arbitrary given degree distribution. It is shown that the results obtained for Erdös-Rényi graphs are generic, at the qualitative level. However, scale-free graphs are an exception to this general rule and exhibit a singular behavior, studied thoroughly in this paper, both analytically and numerically.

I. INTRODUCTION

The best understood random graphs are those with uncorrelated nodes and a local tree structure. It was natural to develop the statistical mechanics of random networks starting with this highly idealized picture, as it is natural to begin explaining the properties of gases starting with the ideal ones. However, short loops show up in most real networks relatively frequently and it is evident that current models should be upgraded to capture this common trait (the present state of art in network research is excellently reviewed in Refs. [1–3]). A possible strategy consists in adding to the Hamiltonian a term favoring the appearance of specific motifs. This strategy was adopted in our recent publication [4], where Erdös-Rényi graphs were perturbed by adding to the action a term proportional to the number of triangles [5]. The purpose of the present paper is to outline a possible generalization of the discussion of Ref. [4] to the case where in zeroth order the graphs belong to the statistical ensemble of simple (i.e., nondegenerate) graphs with uncorrelated nodes and an *arbitrary* degree distribution p_k.

The partition function of the perturbed model is written in the form

$$Z = \sum_M \delta(\mathrm{Tr}(M^2) - 2L)e^{S(M)}\prod_{j=1}^N (p_{k_j}k_j!) = Z_0\langle e^{S(M)}\rangle \quad (1)$$

where N and L are the number of nodes and links, respectively, the sum is over adjacency matrices M, k_j is the degree of the jth node satisfying $\mathrm{Tr}(M^2)=\Sigma_j k_j$, and $S(M)$ is the perturbing Hamiltonian. Setting $S(M)=0$, one obtains a standard model of networks with uncorrelated nodes and the degree distribution p_k (provided the relation $\Sigma_k k p_k/\Sigma_k p_k=2L/N$ holds, see for example Refs [6,7]). In the second line, we denote by $\langle\cdots\rangle$ the average taken in the unperturbed ensemble and therefore Z_0 is, obviously, the partition function of the unperturbed model, in our context an irrelevant overall factor (it can be calculated analytically in the large N limit, see, e.g., [6], but we do not need this result here).

One can, of course, go over from Eq. (1) to a grand-canonical ensemble, with fluctuating L, multiplying Eq. (1) by an appropriate L-dependent weight and summing over L.

In the Erdös-Rényi theory, there is a very natural and simple recipe for the weight. Hence, in Ref. [4], we were grand canonical. Here, we find it more elegant to keep the number of links fixed. Notice, that we are interested in the large N limit where, for any reasonable choice of the weight, L stays close to its average value as a consequence of the constraint $L=1/2\Sigma_j k_j$ and of the central limit theorem.

As in Ref. [4], we set for definiteness $S(M)=G/3!\,\mathrm{Tr}(M^3)$. Expanding the exponential in Eq. (1), we obtain a perturbative series representing the partition function. This perturbative representation was thoroughly studied in Ref. [4] and the analytic arguments were completed by numerical simulations. Let us recall the salient conclusions of this study.

We have introduced a diagrammatic representation of possible contributions to the perturbative expansion, reminiscent of Feynman diagrams used in field theory. Each diagram is a specific subgraph of the full random graph. We have shown that the number of contributing diagrams grows factorially with the order of the perturbation theory.

As is well known, such a factorial growth of the number of diagrams signals a breakdown of the perturbation theory. Indeed, our numerical simulations indicate the presence of a transition from a smooth perturbative regime to a crumpled phase, where almost all nodes form a complete clique [8]. The transition point $G_{out}(N)$ scales like $\ln N$. The two phases are separated by a "barrier" which becomes impenetrable when $N\rightarrow\infty$.

Remarkably enough, it is possible to sum up the leading diagrams, i.e., those whose contribution survives in the limit $N\rightarrow\infty$. Thus, for example, we have been able to derive a closed analytic expression for the average number of triangles. It turns out, that at large enough N this analytic formula is a very good approximation in the almost whole region $G<G_{out}(N)$. Setting $G=G_0\ln N<G_{out}(N)$, one obtains a network model with a nontrivially behaving clustering coefficient $C\propto N^{G_0-1}$. This clustering coefficient is never constant, it falls to zero as $N\rightarrow\infty$, but this fall can be made fairly slow by a proper choice of G_0.

The results summarized above are the starting point of the present paper, which is a direct continuation of Ref. [4]. In Sec. II, we extend our diagrammatic rules to general uncor-

related graphs with a given degree distribution. We also show that the results of Ref. [4] continue to hold, only slightly modified, in this generalized setup. The so-called scale-free graphs are the only exception to the generic behavior and require a separate discussion, presented in Sec. III. Analytic results are confronted to numerical simulations in Sec. IV. We conclude in Sec. V.

II. EXTENDING THE DIAGRAMMATIC RULES

The perturbation series is defined as in Ref. [4]. One calculates the successive terms in the expansion

$$\langle e^{(G/3!)\text{Tr}(M^3)} \rangle = \sum_n \frac{G^n}{6^n n!} \langle [\text{Tr}(M^3)]^n \rangle. \qquad (2)$$

This boils down to the calculation of the expectation value of strings like $M_{a_1 a_2} M_{a_2 a_3} \cdots M_{a_{3n-1} a_{3n}}$. A string does not vanish if all matrix elements involved are equal to unity. The only problem is that the same element, or its transpose, can appear several times in the same string and one needs, therefore, to catalogue all possible string structures (cf. Ref. [9]). This is done with the help of diagrams.

Each matrix element M_{ab} is represented by a line segment with endpoints a and b. A string is then represented by a collection of n triangles, possibly glued together. In order to calculate the perturbation series, one has to consider all possible diagrams. The meaning and the construction of diagrams is explained at length in Ref. [4], with the help of explicit examples. We will not repeat this discussion here, referring the reader to the original paper. We wish only to insist on the salient steps.

(a) All nth-order terms of the series are, of course, proportional to the common factor $G^n/6^n n!$.

(b) Every diagram is a subgraph which has to be embedded in the full graph. An nth-order diagram has, say, v vertices and $v \leq 3n$. These v vertices can be identified with graph nodes in $N!/(N-v)! \sim N^v$ manners.

(c) The same diagram topology usually represents a number of distinct strings. The calculation of this number is the relatively difficult part of the game. But it is universal in the sense that it does not depend on the degree distribution. The calculations of Ref. [4] were done for the Erdös-Rényi graph, but hold quite generally.

(d) Finally, one has to find the expectation value of the string corresponding to a given diagram. In Erdös-Rényi theory, this is simple: If a diagram has ℓ edges, then there are ℓ-independent adjacency matrix elements in the string and the expectation value is just p^ℓ, where p is the control parameter equal to $p = \alpha/N$ when the average degree is set to be finite. The case of a general model of uncorrelated graphs with a given degree distribution requires some extra thought. We will use an argument which is not quite original, since it has already been employed by other authors in a somewhat different context (see, for example, Refs. [10–12]):

The probability that nodes a,b are connected is

$$\text{Prob}(a,b) = \frac{k_a k_b}{\langle k \rangle N}, \qquad (3)$$

where k_a (k_b) is the degree of ath (bth) node and $\langle k \rangle = \Sigma_k k p_k / \Sigma_k p_k$. Notice that $\langle k \rangle N$ equals the number of directed links. The probability in question is inversely proportional to the total number of directed links and is proportional to the degrees of the nodes. Eq. (3) holds when the right-hand side is small enough, i.e., at large N. The probability that b is in turn connected to, say, c is however

$$\text{Prob}(b,c|a) = \frac{(k_b - 1)k_c}{\langle k \rangle N}, \qquad (4)$$

because one link emerging from b, the one connecting b to a, is already occupied: Only $k_b - 1$ links are potentially "active". Pursuing the argument one derives the following rules:

(a) a factor $1/(\langle k \rangle N)$ is associated with every edge of the diagram, and

(b) a factor $k_a!/(k_a - m_a)!$ is associated with every vertex, say a, of the diagram. Here, k_a is the degree of the ath graph node and m_a is the degree of the same node regarded as the diagram vertex.

This was for a particular graph. Averaging, one is led to replace

$$k_a!/(k_a - m_a)! \to \langle k_a!/(k_a - m_a)! \rangle, \qquad (5)$$

in the rules given above [13–15].

We have assumed here that node degrees are uncorrelated, which strictly speaking is only true in the limit $N \to \infty$. Even in a so-called uncorrelated network some "kinematic" correlations appear when one imposes the constraint that there are no self and multiple connections between nodes and when k^2/N is not always negligible [7]. The last condition is easily satisfied when the degree distribution is defined on a finite support, but may be jeopardized in scale-free networks.

Let us check that for Erdös-Rényi graphs, one gets the result of Ref. [4]: when the degree distribution is Poissonian the average of a binomial moment is just a power of α:

$$\langle k_a!/(k_a - m_a)! \rangle = \alpha^{m_a}. \qquad (6)$$

Of course, one has

$$\sum_a m_a = 2\ell. \qquad (7)$$

Furthermore, in Erdös-Rényi theory $\langle k \rangle = \alpha$. Hence, the diagram with ℓ vertices gets a factor

$$\alpha^{2\ell}/(\alpha N)^\ell = p^\ell. \qquad (8)$$

This is exactly what one has in the grand-canonical Erdös-Rényi ensemble and also what one expects in our setup in the large N limit.

We are now equipped to calculate the contribution of an arbitrary diagram. For example, in the limit $G \to 0$, the average number of triangles in a graph, the derivative of the free energy with respect to G, equals

$$\langle T \rangle_{G=0} = \frac{1}{6}\left(\frac{\langle k(k-1)\rangle}{\langle k \rangle}\right)^3, \tag{9}$$

because the diagram has three vertices of order two. Hence, each vertex contributes a factor $\langle k(k-1)\rangle$—remember that vertices are independent—while each link contributes a factor $1/\langle k \rangle$. The powers of N cancel, as in the calculation of Ref. [4] and the only difference is that now instead of α appears a ratio of binomial moments of the degree distribution p_k.

Higher-order diagrams call for other binomial moments. It is important to realize that as long as all the moments of p_k are finite—and therefore N independent for large enough network size—the hierarchy of diagrams in the $1/N$ expansion is the same as in Ref. [4]. In particular, the same diagrams are leading. Summing the leading diagrams, i.e., those whose contribution remains finite when $N \to \infty$, one gets

$$\langle T \rangle = \frac{1}{6}\left(\frac{\langle k(k-1)\rangle}{\langle k \rangle}\right)^3 e^G, \tag{10}$$

which generalizes Eq. (36) of Ref. [4]. Higher-order moments (cumulants) of the T distribution are obtained by differentiating the right-hand side of Eq. (10) with respect to G. One finds that the distribution is Poissonian.

The situation changes when the moments of p_k are not necessarily finite. Without much loss of generality, we will limit our attention to the so-called scale-free graphs, i.e., those where p_k falls at large k like a power: $p_k \propto 1/k^\beta$. We also assume that $\langle k \rangle$ is finite: $\beta > 2$.

III. SCALE-FREE GRAPHS

When $p_k \sim a/k^\beta$ at $k \gg 1$, the moments of order larger than $\beta - 2$ diverge. Actually, at large but finite N, the degree distribution of a nondegenerate uncorrelated graph is cut at

$$k_{\max} \propto N^\gamma, \quad \gamma = \min(1/2, 1/(\beta-1)) \tag{11}$$

(see Ref. [7] for a derivation of this result). Hence, higher-order moments of the degree distribution increase like some powers of N and, consequently, the hierarchy of the $1/N$ expansion is modified compared to Ref. [4]. It turns out that three cases require a separate discussion.

A. Case $\beta > 3$

We will prove that the leading diagrams are the same as those considered in Ref. [4], i.e., those where the number of triangle edges is equal to the number of triangle vertices. One can easily see that in these diagrams the triangles can overlap, but otherwise do not touch.

The proof is by induction. First, we observe that the diagram of order $n=1$ (one triangle) is leading and its contribution tends to a constant as $N \to \infty$. Higher-order diagrams can be constructed by adding successive triangles. Suppose that at the nth order, the leading diagrams are those with the number of links equal to the number of triangles. What happens when one adds the $(n+1)$-st triangle? If it is put on top of an existing triangle or if it is isolated, the new diagram scales with N the same way as its ancestor. In general, the
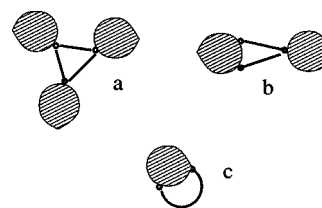

FIG. 1. Additions of a triangle to a diagram, leading to the increase of the number of diagram links by three (a), two (b), and one (c).

new triangle is not isolated and its addition increases the number of links. Three possibilities, illustrated in Fig. 1, can arise. Because of Eq. (11), increasing the degree of a diagram vertex by Δm links produces, at most, an extra divergent factor

$$N^{\Delta m/(\beta-1)}. \tag{12}$$

When the number of links is increased by three [Fig. 1(a)] $\Delta m = 2$ in three vertices. The new links yield a factor N^{-1} each. Thus, the new global factor is at most

$$N^{6/(\beta-1)-3} = N^{(9-3\beta)/(\beta-1)}. \tag{13}$$

Similarly, for Figs. 1(b) and 1(c), one finds that the potentially dangerous new factor is at most

$$N^{(6-2\beta)/(\beta-1)} \quad \text{and} \quad N^{(3-\beta)/(\beta-1)}, \tag{14}$$

respectively. For $\beta > 3$, these factors tend to zero. We conclude that the leading diagrams are those considered in Ref. [4]. The average number of triangles is finite in the limit $N \to \infty$ and is given by Eq. (10).

B. Case $\beta = 3$

As already recalled, at finite N, the degree distribution is cut and the cutoff scales like $\sqrt{N}$. Of course, the cutoff is not sharp, but in calculations it is convenient to replace it by a sharp one $k_{\max} = c\sqrt{N}$. The determination of the constant c is not obvious. We make a choice which yields more or less correct moments of the cut degree distribution: At large enough N
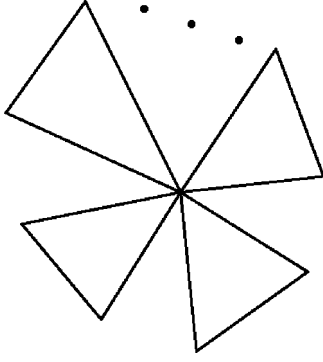
$$\langle k^2 \rangle \approx \frac{a}{2}\ln N \tag{15}$$

and

$$\langle k^m \rangle \approx \frac{ac^{m-2}}{m-2}N^{(m-2)/2}, \quad m > 2 \tag{16}$$

(the sharp cutoff upsets the normalization, but the effect is negligible for large N). The constant c can be roughly estimated using Eq. (16):

$$c \approx \frac{\left(\dfrac{m-2}{a}\langle k^m \rangle\right)^{1/(m-2)}}{\sqrt{N}}. \tag{17}$$

Let us calculate the leading N dependence of an arbitrary diagram with ℓ edges and v vertices: One has a global factor (we omit logs)

FIG. 2. Leading diagram for $\beta=3$.

$$N^{v-\ell+1/2\Sigma_a(m_a-2)}, \tag{18}$$

since each link yields N^{-1}, the v vertices yield N^v and the binomial moments associated with v vertices yield the sum in the exponent. However, this sum can be calculated: $\Sigma_j(m_j-2)=2\ell-2v$. Hence, the total exponent is zero, only logs remain.

As explained in Ref. [4], calculating the average number of triangles at large N it is sufficient to consider connected diagrams only. At each order of perturbation theory, the leading connected diagram is the one with the largest number of vertices of order 2, i.e., the diagram shown in Fig. 2. In the nth order, we have n triangles and therefore the number of vertices of order 2 is $2n$. The central vertex has order $2n$. Each triangle in Fig. 2 can be glued to the central vertex in three manners. The contribution of the nth-order diagram to the free energy is therefore (for $n>1$):

$$\frac{G^n}{6^n n!} 3^n \frac{ac^{2n-2}}{2n-2} \frac{1}{\langle k\rangle^{3n}} \langle k(k-1)\rangle^{2n}$$

$$\sim \frac{G^n}{2(n-1)n!} \frac{a^{2n+1}c^{2n-2}}{(8\langle k\rangle^3)^n} \ln^{2n} N, \tag{19}$$

where we have used the asymptotic result

$$\langle k(k-1)\rangle \sim \frac{a}{2} \ln N. \tag{20}$$

In this limit, Eq. (9) reads

$$\langle T\rangle_{G=0} \sim \frac{a^3}{48\langle k\rangle^3} \ln^3 N. \tag{21}$$

Summing all the connected diagrams one gets the free energy. Differentiating the latter with respect to G one finds the average number of triangles, which can be written in the form

$$\langle T\rangle = T_0 + \frac{a(6T_0)^{2/3}}{4\langle k\rangle} \sum_{n=1}^{\infty} \frac{(bGT_0^{2/3})^n}{nn!} \underset{T_0\to\infty}{\sim} \frac{a}{2c^2 G} e^{bGT_0^{2/3}} \tag{22}$$

where $T_0 \equiv \langle T\rangle_{G=0}$ and

$$b = \frac{c^2 6^{2/3}}{2\langle k\rangle}. \tag{23}$$

It is evident that the increase of $\langle T\rangle$ with $GT_0^{2/3} \propto G \ln^2 N$ is nearly exponential.

C. Case $2<\beta<3$

For $\beta<3$ all the moments, apart from the first, diverge like a power of N. As shown in [7] the cutoff of the degree distribution scales again like $\sqrt{N}$. Therefore, an arbitrary diagram's contribution is proportional to the factor

$$N^{v-\ell+1/2\Sigma_j(m_j-\beta+1)} = N^{v/2(3-\beta)}. \tag{24}$$

Hence, the leading diagrams are those with a maximal number of vertices. One can calculate the number of triangles $\langle T\rangle$ in the leading-power approximation. Again, it is sufficient to consider connected diagrams only. Assuming here, for simplicity of writing, that $\langle k^m\rangle = c^m N^{1/2(m+1-\beta)}$ and using the combinatorial coefficients calculated in Ref. [4], we get:

$$\langle T\rangle = \frac{1}{6} \frac{c^3}{\langle k\rangle^3} N^{3/2(3-\beta)} + \frac{1}{4} \frac{Gc^5}{\langle k\rangle^6} N^{5/2(3-\beta)} + \frac{7}{16} \frac{G^2 c^7}{\langle k\rangle^9} N^{7/2(3-\beta)}$$

$$+ \cdots . \tag{25}$$

It is obvious that the result is very sensitive to the value of c. This would also be true if we were using a more realistic ansatz, viz. the analogue of Eq. (16). A much more reliable prediction is the approximate scaling law:

$$\langle T\rangle = T_0 f(GT_0^{2/3}). \tag{26}$$

In the large N limit, Eq. (9) reads

$$T_0 \equiv \langle T\rangle_{G=0} \propto N^{3/2(3-\beta)}, \tag{27}$$

and $f(x):f(0)=1$ is a positive monotonically increasing function. Although we have calculated only the first few terms in the expansion (25), an educated guess is that the rise of the right-hand side is again nearly exponential.

IV. NUMERICAL RESULTS

In this section, we continue the study of the preceding one but having recourse to Monte Carlo simulations. As in Ref. [4], we use the algorithm of Ref. [7], which has the advantage of generating not only nondegenerate graphs but also thermal fluctuations. The latter point is important because our main goal is to check the stability of the smooth perturbative phase. We will also compare the numerical Monte Carlo data to the predictions of the preceding section, obtained by summing leading diagrams. The agreement will be only semiquantitative, because of large finite-size corrections.

Actually, three types of effects occur at finite N: First, Eqs. (3)–(5) are only approximate, especially when the degree distribution has a fat tail, as already explained. Second, nonleading diagrams are negligible, at fixed G, only asymptotically and it turns out that asymptopia is hard to reach. Furthermore, the nonleading contribution blows up as one approaches the transition point $G=G_{out}$, because the transi-

TABLE I. The average number of triangles at $G=0$: $\langle T \rangle_{G=0}$ from Monte Carlo simulation (left) and estimated from degree distributions using Eq. (9) (right).

N	$\beta=2.5$	$\beta=3$	$\beta=4$
1024	26.36(9) 35.49	11.01(3) 13.15	4.91(2) 5.31
2048	42.26(18) 54.91	15.10(7) 17.58	5.80(2) 6.14
4096	67.05(36) 84.16	20.26(12) 22.93	6.58(3) 6.85
8192	106.2(9) 129.5	27.22(28) 30.28	7.33(3) 7.56
16384	167.0(1.2) 198.9	35.54(33) 38.80	8.01(9) 8.19
$N \to \infty$	$\dfrac{c^3}{48} N^{3/4}$	$\dfrac{1}{6} \ln^3 N$	$\dfrac{32}{3}$

tion is an intrinsically nonperturbative phenomenon. These two effects are in a sense a nuisance for us, since they just obscure the picture. Third, there are manifestations of the fat tail $p_k \sim k^{-\beta}$ in the node degree distribution that persist at any N and are, in fact, increasingly important as N grows up. They are responsible for the singular behavior of scale-free networks and are therefore of much physical interest.

As in our earlier publications we assume, for definiteness, that the degree distribution has the form (cf. Ref. [16])

$$p_k = (\beta-1) \frac{\Gamma(2\beta-3)\Gamma(k+\beta-3)}{\Gamma(\beta-2)\Gamma(k+2\beta-3)} \propto k^{-\beta}(k \gg 1) \quad (28)$$

Similar results were obtained with other choices of p_k.

In Table I is given the average number of triangles at $G=0$, viz. $\langle T \rangle_{G=0}$, measured directly in Monte Carlo simulation (the left column) and estimated using Eq. (9) (the right column) from the degree distribution generated in the same simulation for $\beta=2.5$, 3, and 4, respectively. The agreement improves as N increases and worsens, as expected, with decreasing β and growing the fat tail. Notice, that $\langle T \rangle_{G=0}$ increases with N also for $\beta=4$, although it is expected to be finite in the limit $N \to \infty$: The second moment of the degree distribution is still rising significantly in the explored range of N.

In Fig. 3, we plot $\langle T \rangle / T_0$ versus G for $\beta=4$ and N ranging from 2^{10} to 2^{14}. The asymptotic expectation $\exp(G)$ is also drawn. The data scale and follow the curve $\exp(G)$ at small G. The characteristic fan shows up as one moves toward the transition point $G=G_{out}$. Notice, that largest N data tend to be closer to the curve, as expected. At this point, we are unable to tell how G_{out} behaves. We recall that $G_{out} \propto \ln N$ when in
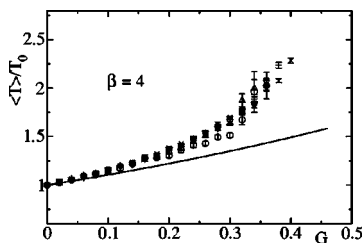
zeroth order the graphs are of Erdös-Rényi type. This is expected to be the generic behavior when the degree distribution effectively has a finite support. In this respect, the status of the $\beta=4$ case is uncertain. There is no evidence in the data for an increase of G_{out} with N. The apparent constancy of G_{out} could be a finite-size effect, however. On the other hand, it is very plausible that hubs, i.e., nodes with largest degree, behave as seeds of Strauss cliques, preventing the growth of the barrier separating the smooth phase from the crumpled one. In order to settle this question, one would have to simulate enormous networks, beyond reach with present means.

In Fig. 4, we plot $(\langle T \rangle - T_0)/T_0^{2/3}$ versus $G T_0^{2/3}$, as suggested by Eq. (22), for $\beta=3$ and N ranging again from 2^{10} to 2^{14}. The data scale reasonably well, especially at low $G T_0^{2/3}$. It appears that, very roughly

$$G_{out} \approx \frac{1.5}{T_0^{2/3}} \propto \ln^{-2} N. \quad (29)$$

The constant c can be estimated from Eq. (17), using the observed (i.e., cut) degree distributions. A reliable estimate is obtained from low-order moments and using rather large N data; one finds c ranging from 0.9 to 1.2. Setting $c=0.9$, one gets from Eq. (22) the curve shown in Fig. 4.

In Fig. 5, we plot $\langle T \rangle / T_0$ versus $G T_0^{2/3}$, as suggested by Eq. (25), for $\beta=2.5$ and N ranging from 2^{10} to 2^{14}. One observes the expected scaling at small values of $G T_0^{2/3}$, but at larger values finite-size effects become gradually more and more important. It appears that, very roughly

$$G_{out} \approx \frac{2.3}{T_0^{2/3}} \propto N^{-1/2} \quad (30)$$
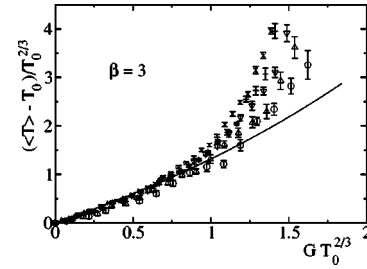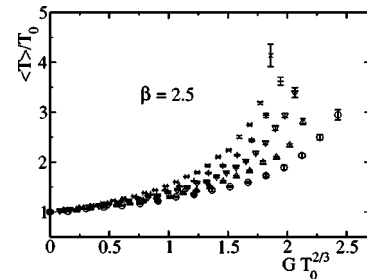

FIG. 4. $(\langle T \rangle - T_0)/T_0^{2/3}$ versus $G T_0^{2/3}$ for $\beta=3$ and $N=2^{10}$ ($\times$), 2^{11} (+), 2^{12} ($\triangledown$), 2^{13} ($\triangle$), 2^{14} ($\bigcirc$).

FIG. 3. $\langle T \rangle / T_0$ versus G for $\beta=4$ and $N=2^{10}$ ($\times$), 2^{11} (+), 2^{12} ($\triangledown$), 2^{13} ($\triangle$), 2^{14} ($\bigcirc$).

FIG. 5. $\langle T \rangle / T_0$ versus $G T_0^{2/3}$ for $\beta=2.5$ and $N=2^{10}$ ($\times$), 211 (+), 2^{12} ($\triangledown$), 2^{13} ($\triangle$), 2^{14} ($\bigcirc$).
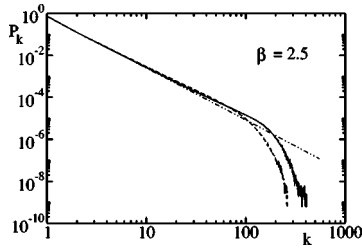
FIG. 6. The degree distribution P_k at $N=2^{14}$ and for $\beta=2.5$. Dashed line is for $G=0$, solid line for $G=0.08$ (just before the transition to the crumpled phase). The almost straight line corresponds to the asymptotic shape of P_k, i.e., to p_k.

In Fig. 6, we plot the degree distribution at $N=2^{14}$ and $\beta=2.5$, for $G=0$ and 0.08 (just before the transition to the crumpled phase), to show that the introduction of the interaction Hamiltonian does not spoil the scale free property. In Fig. 7, we show the variation with k of the clustering parameter

$$C_k = \frac{2T_k}{k(k-1)}, \tag{31}$$

where T_k is the average number of triangles touching a vertex with degree k. One observes that the decrease of the clustering parameter is much slower than observed in Internet, for example, where $C_k \propto k^{-0.75}$ [17].

V. SUMMARY AND CONCLUSION

This paper is a direct continuation of Ref. [4], where we have studied perturbed random Erdös-Rényi graphs. In the present paper, we show that the results of Ref. [4] continue to hold when an ensemble of (almost) arbitrary uncorrelated graphs is perturbed by the same interaction, favoring the formation of triangles. There is, however, a notable exception to this generic behavior: The so-called scale-free graphs behave differently, especially when the degree distribution has a diverging variance.

At finite N, the smooth phase exists only when the interaction coupling G is smaller than some threshold $G < G_{\text{out}}$. Generically, G_{out} scales like ln N, but for scale-free networks with $2 < \beta < 3$ it scales like $N^{\beta-3}$ (modulo logs). Hence, the support of the smooth regime does not expand but shrinks to zero in the thermodynamic limit. There is nothing dramatic
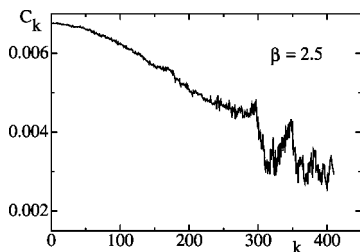

FIG. 7. Clustering coefficient versus node degree at $N=2^{14}$, $\beta=2.5$, and $G=0.08$ (just before the transition to the crumpled phase). Notice, that the scales are linear.

in such a behavior, which is also encountered in more conventional matrix models. It just means that G is not the physical coupling and that the latter is rather $GN^{3-\beta}$, again modulo logs. When the physical coupling is used, the smooth phase lives in a finite coupling interval and the average number of triangles is enhanced compared to the unperturbed expectation. However the stability of this phase is an open problem. In Ref. [4], we found that nonperturbative phenomena are manifestly negligible almost in the whole smooth phase: Numerical data were remarkably close to perturbative predictions. Here, we cannot claim the same, partly because we are unable to disentangle finite-size and nonperturbative effects.

At finite G, scale-free networks are unstable for N large enough. The physical significance of this singular behavior is not fully clear yet. In our simulations, the thermal motion consists of network rewirings. Rewiring is an ergodic move: Every two states can be transformed one into another by making a finite number of rewirings. In particular, any other algorithmic move could be regarded as made up of rewirings. However, with a different algorithm the thermalization time of the system would in general be different, in particular it could explode with increasing N. Thus, it is not excluded, although does not seem very likely, that the instability is an algorithm artifact.

Very many natural networks are scale free, with the exponent β below 3. At least some of them seem fairly stable. In some cases, there are selection rules constraining the rewirings. But this is not the most interesting possibility. How do the natural networks compare to the graphs of our model? We see one significant difference: In our graphs, the clustering coefficient is weakly correlated with node degrees, while in natural networks it tends to decrease like some power of the latter. The behavior of our graphs is easy to understand as follows: The system forms triangles at random and therefore tends to attach many triangles to hubs, which apparently become seeds of Strauss cliques. Natural networks seem to avoid this disease by suppressing the formation of triangles at hubs. It is plausible that a specific hierarchical organization screens natural networks from the instability. In such a scenario, a triangle generating term with a finite coupling G could after all be present in the Hamiltonian. For the moment, this is just a speculation.

Our paper is not a phenomenological one. We are not yet at the stage of constructing a model to be compared to the data. We focus on the theoretical problem of the stability of networks with respect to motif generating terms in the Hamiltonian. However, the paper is not quite devoid of phenomenological implications: Our method allows us not only to calculate averages of physical quantities characterizing an individual network, but also fluctuations of those quantities in the ensemble, giving us an insight into the problem of typicality of networks. As far as we know, the magnitude of fluctuations of motifs has never been estimated for graphs with an arbitrary given degree distribution.

[1] R. Albert and A.-L. Barabasi, Rev. Mod. Phys. **74**, 47 (2002).

[2] S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. **51**, 1079 (2002); *Evolution of Networks: from Biological Nets to to the Internet and WWW* (Oxford University Press, New York, 2003).

[3] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).

[4] Z. Burda, J. Jurkiewicz, and A. Krzywicki, Phys. Rev. E **69**, 026106 (2004).

[5] Similar ideas were independently developed in S. Coulomb, M. Bauer, e-print cond-mat/0310170.

[6] J. Berg and M. Lässig, Phys. Rev. Lett. **89**, 228701 (2002).

[7] Z. Burda and A. Krzywicki, Phys. Rev. E **67**, 046118 (2003); the same model/algorithm was already discussed, but for trees and pseudographs, in Z. Burda, J. D. Correia, A. Krzywicki, *ibid.* **64**, 046118 (2001).

[8] The existence of this crumpled phase was discovered by D. Strauss, SIAM Rev. **28**, 513 (1986); see also P. W. Holland and S. Leinhardt, J. Am. Stat. Assoc. **76**, 33 (1981).

[9] M. Bauer and O. Golinelli, J. Stat. Phys. **103**, 301 (2001).

[10] A. Fronczak, P. Fronczak, and J. A. Holyst, e-print cond-mat/0212230.

[11] B. Söderberg, Acta Phys. Pol. B **34**, 5058 (2003).

[12] S. Itzkovitz *et al.*, Phys. Rev. E **68**, 026127 (2003).

[13] The rules to find the expectation value of the string of a given diagram can be derived elegantly using the hidden variable formulation of network models. Without entering into details, we outline the idea: Let the letter h denote the random hidden variable associated with a node and let the probability of connecting two nodes be $f(h)f(h')/N$. Using the formalism of the model one finds $\langle k!/(k-m)! \rangle = \langle f(h)^m \langle f(h)^m \rangle$ and, in particular, $\langle k \rangle = \langle f(h) \rangle^2$. Using this result and the fact that with each vertex, say a, of the diagram is associated a factor $\langle f(h)^{m_a} \rangle / N^{m_a/2}$, one easily recovers the rules given in the main body of the text.

[14] B. Söderberg, Phys. Rev. E **66**, 066121 (2002).

[15] M. Boguna and R. Pastor-Satorras, Phys. Rev. E **68**, 036112 (2003).

[16] P. L. Krapivsky and S. Redner, Phys. Rev. E **63**, 066123 (2001).

[17] A. Váquez, R. Pastor-Satorras, and A. Vespignani, e-print cond-mat/0106084.